$y_i^L \quad y_i^R \quad i \in I$

$$y_i^L(y_j^R(p)) = y_j^R(y_i^L(p)) \qquad p \in \mathcal{B}(V)$$

$p = x_1 x_2 x_3 x_4 x_5$

$$y_4^L(y_2^R(p)) = y_2^R(y_4^L(p))$$

$$\boxed{g_j \cdot (y_i^L(g_j^{-1} \cdot p)) = q_{ji}^{-1} y_i^L(p)}$$

— — — —

It's a standard fact:

the algebra generated by $\{y_i^L\}$

$\cong \mathcal{B}(V^*)$

$x_i \mapsto y_i^L$ extends to an algebra iso $l : \mathcal{B}(V) \to \mathcal{B}(V^*)$

Finally, by [5]

$\exists!$ bilinear map $\langle \cdot , \cdot \rangle$:

$$\mathcal{B}(V^*) \# \Bbbk G \otimes \mathcal{B}(V) \to \mathcal{B}(V)$$

which defines a $\mathcal{B}(V^*) \# \Bbbk G$ module algebra on $\mathcal{B}(V)$.

3. Finiteness condition

$\exists$ PBW-basis of $\mathcal{B}(V)$

(P) the height of a PBW generator of $\mathbb{Z}^n$-degree d is finite

$\Leftrightarrow 2 \le \operatorname{ord}_\chi(d,d) < \infty$

height $= \operatorname{ord}_\chi(d,d)$

Let $\Delta^+(\mathcal{B}(V))$ denote the set of degrees of the restricted PBW generators of $\mathcal{B}(V)$ counted with multiplicities.

Note that this definition is independent of the choice of a $\mathbb{Z}^n$-graded PBW-basis satisfying property (P)

$\mathcal{B}(V)$ is $\mathbb{Z}^n$-graded

$\Delta^+(\mathcal{B}(V)) \subset \mathbb{N}_0^n$

$\Delta(\mathcal{B}(V)) = \Delta^+(\mathcal{B}(V)) \cup -\Delta^+(\mathcal{B}(V))$

Let's consider the following finite conditions on $\mathcal{B}(V)$

(F1) $\dim_{\Bbbk} \mathcal{B}(V) < \infty$

(F2) $\Delta^+(\mathcal{B}(V))$ is finite

(F3) $\operatorname{Dim}_{\Bbbk}(\mathcal{B}(V)) < \infty$ — GK-dim

(F1) $\Rightarrow$ (F2) obviously

(F2) $\Rightarrow$ (F1) if height $< \infty$

(F2) $\Rightarrow$ (F3) $|\Delta^+(\mathcal{B}(V))| = M$

(F3) $\not\Rightarrow$ (F2) unknown

$\mathcal{B}(V) \cong \Bbbk[x_1, \cdots, x_M]$ as V.S, GK-dim $= M$

Define $\operatorname{ad}_c x_i(p) := x_i p - (g_i \cdot p) x_i$, $p \in \mathcal{B}(V)$

Consider the sets

$$M_{i,j} := \{(\operatorname{ad}_c x_i)^m (x_j) \mid m \in \mathbb{N}_0\}$$

for $i \neq j$, $i,j \in \{1,2,\cdots,n\}$

By [10, lemma 20] if (F3) holds then all $M_{i,j}$ are sets.

Define $m_{ij} := \min\{m \in \mathbb{N}_0 \mid (m+1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0\}$

then

m_{ij} is well defined $\Leftrightarrow$ $M_{i,j}$ is finite.

In this case, $(\operatorname{ad}_c x_i)^{m_{ij}}(x_j) \neq 0$

Fix $i \in \{1, \cdots, n\}$

If $M_{i,j}$ is finite for $\forall j$ (For example (F3) holds)

then one can ~~def~~ introduce a $\mathbb{Z}$-linear mapping s_i:

$$\mathbb{Z}^n \to \mathbb{Z}^n$$

$$s_i(e_j) := \begin{cases} -e_i & j = i \\ e_j + m_{ij} e_i & \text{if } j \neq i \end{cases}$$

$s_i^2 = \mathrm{id}$

4. Transformations of Nichols algebras

$\mathcal{B}(V) \to \mathcal{B}(V_i)$
$\mathcal{B}_i$

Assume $\mathcal{B}(V)$ is rank n, diagonal type

Suppose that $i \in \{1, \cdots, n\}$ s.t. $M_{i,j}$ is finite, $\forall j$

We describe the construction of a Nichols algebra associated to i.

Construct the smash product $\qquad \mathbb{Z}e_i$

$$H_i := \underbrace{\Bbbk[y_i^R]}_{\text{is a braided hopf algebra in } {}^{\Bbbk[e_i,e_i^{-1}]}_{\Bbbk[e_i,e_i^{-1}]}\mathcal{YD}} \# \underbrace{\Bbbk[e_i, e_i^{-1}]}$$

It has a unique Hopf algebra structure satisfying: $e_i y_i^R = q_{ii}^{-1} y_i^R e_i$ ✓

$\Delta(e_i) = e_i \otimes e_i$ ✓.

$\Delta(y_i^R) = 1 \otimes y_i^R + y_i^R \otimes e_i^{-1}$

$R \# H \cong R \otimes H$ as V.S

$(r \# h)(s \# f) = r(h_1 \cdot s) \# h_2 f$

$\Delta(r \# h) = r^{(1)} \# r^{(2)}{}_{(-1)} h_1 \otimes r^{(2)}{}_{(0)} \# h_2$

R型的余乘

$\Delta_R(y_i^R) = y_i^R \otimes 1 + 1 \otimes y_i^R$

$\Delta(y_i^R) = y_i^R \otimes 1 + e_i^{-1} \otimes y_i^R$ ($\delta(y_i^R) = e_i^{-1} \otimes y_i^R$)

By equation (10) and (8)

$\mathcal{B}(V)$ is an H_i-module algebra

module $\qquad$ module algebra

e_i, e_i^{-1} act via $\triangleright$

y_i^R acts by evaluation

$H_i \ \triangleright$

Let $\left((\mathcal{B}(V))^{op} \# H_i^{cop}\right)^{op}$

denote the opposite algebra of the smash product of $\mathcal{B}(V)^{op}$ and H_i^{cop}

Note that it contains $\mathcal{B}(V)$ and $H_i^{op,cop}$ as subalgebras.

and one has $\boxed{ph = h_1(h_2 \triangleright p)}$

$\Delta(h) = h_1 \otimes h_2$ denote the coproduct of $h \in H_i$

in particular

$$p y_i^R = y_i^R \cdot (e_i^{-1} \triangleright p) + y_i^R(p)$$

$$\begin{aligned} ph &= (p \# 1) \cdot (1 \# h) \\ &= (1 \# h)(p \# 1) \\ &= h_2 \triangleright p \# h_1 \\ &= (h_2 \triangleright p \# 1)(1 \# h_1) \\ &= h_1 (h_2 \triangleright p) \end{aligned}$$

Further $\boxed{\left(\mathcal{B}(V)^{op} \# H_i^{cop}\right)^{op}}$ is a Y-D module over $\Bbbk G$ where $\cdot$, δ is defined by:

$\delta(x_j) = g_j \otimes x_j \quad \delta(y_i^R) = g_i^{-1} \otimes y_i^R$.

— — — —